Lecture 13 Summary

Tues 19/08/14

Vocabulary

* Cauchy sequence
* complete metric space
* isometry
* completion

Examples

NON CONVERGENT SEQUENCES

(1) $X = (0,1]$ with $d(x,y) = |y-x|$ and $\vec{x} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$.
There is no limit for $\vec{x}$ since $0 \notin X$.

(2) $\vec{x} = (3, 3.1, 3.14, 3.1415, 3.14159, 3.141592, \dots)$ is a sequence in $\mathbb{Q}$.
$\vec{x}$ does not converge in $\mathbb{Q}$.

COMPLETE & NOT COMPLETE METRIC SPACES

(1) $\mathbb{R}$ with metric $d(x,y) = |y-x|$ is complete.
(2) $\mathbb{Q}$ with metric $d(x,y) = |y-x|$ is not complete.
(3) As a metric space, $\mathbb{C} = \mathbb{R}^2$ and so $\mathbb{C}$ is complete.

ISOMETRY THAT IS NOT SURJECTIVE

$$\varphi : \mathbb{Q} \to \mathbb{R}$$
$$x \mapsto x$$

Homework

* If $x_1, x_2, \dots$ is a convergent sequence then $x_1, x_2, \dots$ is a Cauchy sequence.

* Give an example of a Cauchy sequence that is not a convergent sequence.

* Let (X,d) be a metric space and let $Y \subseteq X$. Show that if X is complete and Y is closed then Y is complete.

* Let (X,d) be a metric space and let $Y \subseteq X$. Show that if Y is complete then Y is closed.

* Show that $\mathbb{R}$ (with the standard metric) is complete.

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Homework (Continued)

- Show that if $X_1, X_2, \ldots, X_m$ are complete metric spaces, then $X_1 \times X_2 \times \ldots \times X_m$ is a complete metric space. (In particular, $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$ is complete.)

- Let X and Y be metric spaces and let
$$C_b(X, Y) = \{f : X \to Y \mid f \text{ is continuous and bounded}\}$$
with norm $\rho : C_b(X,Y) \times C_b(X,Y) \to \mathbb{R}_{\geq 0}$ given by
$$\rho(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$
Show that if Y is complete then $C_b(X,Y)$ is complete. (In particular, $C_b(X, \mathbb{R})$ is complete.)

- Show that if $\varphi : X \to Y$ is an isometry then φ is injective.

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